ABSTRACT

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MEAN NUMBER AND VARIANCE OF THE NUMBER OF REAL ZEROS OF A RANDOM TRIGONOMETRIC POLYNOMIAL

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Suppose that the real numbers a_k ($k = 1, 2, \ldots, n$) are independent random variables, each of which is normally distributed with mean 0 and variance 1. We define the random trigonometric cosine polynomial $f(x)$ so that

$$f(x) = a_1 \cos x + a_2 \cos 2x + \ldots + a_n \cos nx.$$

Let N be the number of zeros of $f(x)$ in the real interval $(0, 2\pi)$. Then N is itself a random variable. An explicit formula for the mean value of N can be found in the literature. An asymptotic expansion of this formula, valid for large values of n, has also been derived. On the other hand, relatively little is known about the variance of N. Farahmand has a result that implies a formula for the variance of N when $n \geq 4$. He has used his result to show that the variance $O(n^{3/2})$ for large n. This is the best result known at present. We will calculate numerical values for the variance of N when n is small, i.e., $n = 1, 2$, and 3.

MEAN NUMBER AND VARIANCE OF THE NUMBER OF REAL ZEROS OF A RANDOM TRIGONOMETRIC POLYNOMIAL

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CHAPTER 1

Introduction

Suppose that $a_1, a_2, \ldots, a_n$ are independent, normally distributed random variables each with mean 0 and variance 1. The number of real zeros of the random trigonometric cosine polynomial,

$$f(x) = a_1 \cos x + a_2 \cos 2x + \ldots + a_n \cos nx,$$

in the real interval $(0, 2\pi)$ is the random variable N. An explicit formula for the mean value ν_n of N can be found in the literature [2, 3, 8, 9]. An asymptotic expansion of this formula, valid for large values of n, has also been derived [9]. On the other hand, little is known about the variance σ_n^2 of N. Farahmand [5] has a result when $n \geq 4$, and he has deduced an upper bound for σ_n^2. He later [6] obtained a better result that states that σ_n^2 is $O(n^{3/2})$ for large n. This is the best result known at present. Summaries of much of the information can be found in one or both of the monographs [2, 7].

We will calculate numerical values for the variance of N when n is small, i.e. $n = 1$, 2, and 3. The calculation is based on an initial explicit determination of N as a function of the random coefficients, followed by an appropriate integration of N and N^2 over the probability distribution of the coefficients. This approach is feasible when n is small although possibly unsuitable for large values of n. Integral formulas for the expected values, ν_n of N and μ_n of N^2, when n is arbitrary, can be found in Chapter 2. Then the variance of N is $\sigma_n^2 = \mu_n - \nu_n^2$. In Chapters 3 and 4 we apply these formulas to the special cases $n = 1$ and $n = 2$, and $n = 3$, respectively. After some numerical analysis discussed in Chapter 5, we obtain the results for ν_3 and σ_3^2 that are tabulated in Table 1.

Table 1. MEAN AND VARIANCE OF N

n	ν_n	σ_n^2
1	2	0
2	3	1
3	4.1578	2.4078

CHAPTER 2

The General Case

Let $N(a_1, a_2, . . . , a_n)$ be the number of zeros on the real interval $(0, 2\pi)$ of the random trigonometric polynomial,

$$f(x) = a_1 \cos x + a_2 \cos 2x + ... + a_n \cos nx,$$

in which the coefficients a_k $(k = 1, . . . , n)$ are independent real valued random variables, each normally distributed with mean 0 and variance 1, i.e., the probability density function for a_k is $(2\pi)^{-\frac{1}{2}} \exp(-\frac{1}{2}a_k^2)$. Then N and N^2 are random variables whose respective expected values are the following:

$$\nu_n = (2\pi)^{-n/2} \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} \exp\{-\frac{1}{2}(a_1^2 + a_2^2 + ... + a_n^2)\} N(a_1, ..., a_n) da_1 ... da_n,$$

$$\mu_n = (2\pi)^{-n/2} \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} \exp\{-\frac{1}{2}(a_1^2 + a_2^2 + ... + a_n^2)\} N^2(a_1, ..., a_n) da_1 ... da_n.$$

Moreover, the variance σ_n^2 of the random variable N is $\mu_n - \nu_n^2$.

CHAPTER 3

The Cases When $n = 1$ and $n = 2$

When $n = 1$, the number of real zeros of the random trigonometric polynomial

$$f(x) = a_1 \cos x$$

on the interval $(0,\ 2\pi)$ is 2. If ν_1 is the expected value of $N(a_1)$, then $\nu_1 = 2$. Moreover, $\mu_1 = 4$ and $\sigma_1^2 = 0$.

When $n = 2$, let $N(\alpha, \beta; a_1, a_2)$ be the number of real zeros of the random trigonometric polynomial

$$f(x) = a_1 \cos x + a_2 \cos 2x$$

on the interval $(\alpha,\ \beta)$. If ν_2 is the expected value of $N(0, 2\pi; a_1, a_2)$, then

$$\nu_2 = (1/2\pi) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} N(0, 2\pi; a_1, a_2) \exp\{-\frac{1}{2}(a_1^2 + a_2^2)\} da_1 da_2.$$

Because $f(x) = f(2\pi - x)$, we see that $N(0, 2\pi; a_1, a_2) = 2N(0, \pi; a_1, a_2)$. Hence,

$$\nu_2 = (1/\pi) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} N(0, \pi; a_1, a_2) \exp\{-\frac{1}{2}(a_1^2 + a_2^2)\} da_1 da_2.$$

Moreover, $N(0, \pi; a_1, a_2)$ is an even function of both a_1 and a_2. Therefore,

$$\nu_2 = (4/\pi) \int_{0}^{\infty} \int_{0}^{\infty} N(0, \pi; a_1, a_2) \exp\{-\frac{1}{2}(a_1^2 + a_2^2)\} da_1 da_2.$$

Let $y = \cos x$, so that $\cos 2x = 2y^2 - 1$. Then

$$f(x) = 2a_2y^2 + a_1y - a_2,$$

so that $f(x)$ becomes a function $g(y)$.

When $a_1 > 0$ and $a_2 > 0$, it is clear that $g(0) = -a_2 < 0$, $g(1) = a_1 + a_2 > 0$. Hence, there is a unique y_1 such that $0 < y_1 < 1$ and $g(y_1) = 0$. Because the product of the zeros of $g(y)$ is -$\frac{1}{2}$, the other zero of $g(y)$ is $y_2 = -1/(2y_1)$. This zero is negative because y_1 is positive, and is in the interval (-1,0) if and only if $y_1 > \frac{1}{2}$. Note that $g'(y) = 4a_2y + a_1 > 0$ when $y > 0$, so that $y_1 > \frac{1}{2}$ if and only if $g(\frac{1}{2}) < 0$. Also note that $g(\frac{1}{2}) = (a_1 - a_2)/2 < 0$ if and only if $a_2 > a_1$. Therefore, $N(0, \pi; a_1, a_2) = 2$ when $a_2 > a_1 > 0$, and $N(0, \pi; a_1, a_2) = 1$ when $a_1 > a_2 > 0$. Hence,

$$\nu_2 = (8/\pi)\int_0^\infty [\int_{a_1}^\infty e^{-a_2^2/2}da_2]e^{-a_1^2/2}da_1 + (4/\pi)\int_0^\infty [\int_0^{a_1} e^{-a_2^2/2}da_2]e^{-a_1^2/2}da_1.$$

Suppose that $\rho = \int_{a_1}^\infty e^{-a_2^2/2}da_2$, $w = \int_0^{a_1} e^{-a_2^2/2}da_2$, so that $\rho(0) = w(\infty) = \eta = \sqrt{\pi/2}$. Then

$$\pi\nu_2 = 8\int_0^\eta \rho\, d\rho + 4\int_0^\eta w\, dw = 6\eta^2 = 3\pi,$$

so that $\nu_2 = 3$. Similarly,

$$\mu_2 = (8/\pi)\int_0^\infty \int_0^\infty N^2(0, \pi; a_1, a_2)\exp\{-\frac{1}{2}(a_1^2 + a_2^2)\}da_1da_2,$$

$$\pi\mu_2 = 32\int_0^\infty [\int_{a_1}^\infty e^{-a_2^2/2}da_2]e^{-a_1^2/2}da_1 + 8\int_0^\infty [\int_0^{a_1} e^{-a_2^2/2}da_2]e^{-a_1^2/2}da_1$$

$$= 32 \int_0^{\eta} \rho \, d\rho + 8 \int_0^{\eta} w \, dw = 20\eta^2 = 10\pi,$$

so that $\mu_2 = 10$. Therefore, variance σ_2^2 of the random variable N is $\mu_2 - \nu_2^2 = 1$.

CHAPTER 4

The Case When $n = 3$

When $n = 3$, the polynomial to be considered on the interval $(0, 2\pi)$ is

$$f(x) = a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x.$$

Let $N(\alpha, \beta; a_1, a_2, a_3)$ be the number of zeros of $f(x)$ on the real interval (α, β). If ν_3 is the expected value of $N(0, 2\pi; a_1, a_2, a_3)$, then

$$\nu_3 = (1/2\pi)^{3/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} N(0, 2\pi; a_1, a_2, a_3) \exp\{-\frac{1}{2}(a_1^2 + a_2^2 + a_3^2)\} da_1 da_2 da_3.$$

Because $f(x) = f(2\pi - x)$, we see that $N(0, 2\pi; a_1, a_2, a_3) = 2N(0, \pi; a_1, a_2, a_3)$. Consequently,

$$\nu_3 = 2^{-1/2}\pi^{-3/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} N(0, \pi; a_1, a_2, a_3) \exp\{-\frac{1}{2}(a_1^2 + a_2^2 + a_3^2)\} da_1 da_2 da_3.$$

Let $y = \cos x$, so that $\cos 2x = 2y^2 - 1$ and $\cos 3x = 4y^3 - 3y$. Then $f(x) = 4a_3 G(y; u, v)$, in which $G(y; u, v) = y^3 + 2vy^2 + uy - v$, $u = (a_1 - 3a_3)/(4a_3)$, and $v = a_2/(4a_3)$. Let $N(u, v)$ be the number of real zeros of $G(y; u, v)$ on the interval $(-1, 1)$, so that

$$N(u, v) = N(0, \pi; a_1, a_2, a_3) \text{ if } a_3 \neq 0,$$

$$\nu_3 = 2^{-1/2}\pi^{-3/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} N(u,v)\exp\{-\frac{1}{2}(a_1^2+a_2^2+a_3^2)\}da_1da_2da_3.$$

In order to justify the last equation, we observe that the set of triples (a_1, a_2, a_3) for which $a_3 = 0$ is a set of measure zero in the Euclidean space of all triples.

We wish to change the variables of integration from a_1, a_2 to u, v. For this purpose we need the Jacobian

$$\begin{vmatrix} \partial a_1/\partial u & \partial a_1/\partial v \\ \partial a_2/\partial u & \partial a_2/\partial v \end{vmatrix}.$$

Because $a_1 = (4u+3)a_3$ and $a_2 = 4a_3$v, the Jacobian is $16a_3^2$. Moreover, $a_1^2+a_2^2+a_3^2 = 2M(u,v)$, in which $M(u,v) = 8u^2+8v^2+12u+5$. Therefore,

$$\nu_3 = 2^{-1/2}\pi^{-3/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} N(u,v)\exp\{-a_3^2M(u,v)\}16a_3^2dudvda_3,$$

$$\nu_3 = 2^{7/2}\pi^{-3/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} N(u,v)\int_{-\infty}^{\infty}\exp\{-a_3^2M(u,v)\}a_3^2da_3dudv.$$

It is known [1, p. 302, 7.4.4] that $\int_{-\infty}^{\infty}\omega^2\exp(-M\omega^2)\,d\omega = 2^{-1}M^{-3/2}\pi^{1/2}$. Hence

$$\nu_3 = 2^{5/2}\pi^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} N(u,v)M(u,v)^{-3/2}dudv.$$

Because $G(-y;u,-v) = -G(y,u,v)$, we see that N is an even function of v, i.e., $N(u,-v) = N(u,v)$. Therefore,

$$\nu_3 = 2^{7/2}\pi^{-1}\int_{-\infty}^{\infty}du\int_{0}^{\infty} N(u,v)M(u,v)^{-3/2}dv,$$

and the expected value μ_3 of N^2 is such that

$$\mu_3 = 2^{9/2}\pi^{-1} \int_{-\infty}^{\infty} du \int_0^{\infty} N^2(u,v) M(u,v)^{-3/2} dv.$$

We observe the following particular values of G:

$$\begin{aligned} G(-\infty,\ u,\ v) &= -\infty, \\ G(-1,\ u,\ v) &= v-u-1, \\ G(0,\ u,\ v) &= -v<0, \\ G(1,\ u,\ v) &= v+u+1, \\ G(+\infty,\ u,\ v) &= +\infty. \end{aligned}$$

Descartes' rule of signs [4, p. 77] implies that there exists a unique positive y_1 such that $G(y_1,u,v) = 0$. This y_1 lies in the interval $(0,1)$ if and only if $G(1,u,v) = v+u+1 > 0$, i.e., $u > -v-1$. Because $G(0,u,v) = -v < 0$, there exists a unique y_2 such that $-1 < y_2 < 0$ and $G(y_2,u,v) = 0$ if and only if $G(-1,u,v) = v-u-1 > 0$, i.e., $u < v-1$. We can now observe that $N(u,v) = 2$ if $-v-1 < u < v-1$; the two zeros are y_1 and y_2.

If $u < -v-1$, then $G(1,u,v) < 0$, which implies that $y_1 > 1$. Hence, $G(y,u,v)$ has no zeros on $(0,1)$. Moreover, $u < v-1$ when $u < -v-1$. Hence, there exists a unique y_2 such that $G(y_2,u,v) = 0$ and $-1 < y_2 < 0$. Therefore, $N(u,v) = 1$ when $u < -v-1$.

Now suppose that $u > v-1$ and $-1 < y < 0$. Then $G(y,u,v) < y^3 + 2vy^2 + (v-1)y - v = (y+1)[y^2 + (2v-1)y - v]$. When $y = 0$, $y^2 + (2v-1)y - v = -v < 0$.

Also when $y = -1, y^2 + (2v-1)y - v = 2 - 3v$. We observe that the quadratic polynomial, $y^2 + (2v-1)y - v$ is a convex function of y and its values at the endpoints of the interval $(-1, 0)$ are respectively the numbers $2-3v$ and $-v$, both of which are negative when $v > 2/3$. Therefore, $y^2 + (2v-1)y - v < 0$ when $-1 \leq y \leq 0$ and $v > 2/3$. Hence, $G(y,u,v) < 0$ when $-1 \leq y \leq 0$ and $v > 2/3$, and $G(y,u,v)$ has no zeros on the interval $(-1,0)$. Because $u > -v-1$ when $u > v-1$, we know that there is a unique zero y_1 of G in the interval $(0,1)$. Therefore, $N(u,v) = 1$ when $u > v-1$ and $v > 2/3$.

Next, suppose that $u > v-1$ and $0 < v < 2/3$. We observe that

$$9(\partial G/\partial y) = 9(3y^2 + 4vy + u) = 3(3y+2v)^2 + 4(3u - 4v^2).$$

Hence, $(\partial G/\partial y) > 0$ for all y if $u > (4v^2)/3$. Consequently, the maximum of $G(y,u,v)$ on the interval $(-1,0)$ is $G(0,u,v) = -v < 0$. Therefore, $G(y,u,v) < 0$ when $-1 \leq y \leq 0$ i.e., there are no negative zeros of G on $(-1,0)$. However, there does exist a unique y_1 such that $0 < y_1 < 1, G(y_1,u,v) = 0$. Therefore, $N(u,v) = 1$ when $u > v-1, 0 < v < 2/3$, and $u > (4v^2)/3$. Of course, $(4v^2)/3 > v-1$ for all real v, so that the inequality, $u > v-1$, is superfluous.

Finally, suppose that $0 < v < 2/3$ and that $v - 1 < u < (4v^2)/3$. Then $\partial G/\partial y = 3y^2 + 4vy + u = 0$ when $y = y_0$, in which

$$y_0 = (-2v \pm \sqrt{4v^2 - 3u})/3.$$

Moreover, $(\partial^2 G)/(\partial y^2) = 6y + 4v = \pm\, 2\sqrt{4v^2 - 3u}$ when $y = y_0$. Therefore, G has a local maximum when the minus sign is used, so that $3y_0 = -2v - \sqrt{4v^2 - 3u} < 0$. Moreover, $y_0 > -1$ if and only if $\sqrt{4v^2 - 3u} < 3 - 2v$. Because $3 - 2v > 0$ when $v < 2/3$, the last inequality holds if and only if $4v^2 - 3u < (3-2v)^2$, or if and only if $u > 4v - 3$. This inequality is surely true because $u > v - 1 > 4v - 3$ when $v < 2/3$. The local maximum value of G is the maximum value of G on $(-1, 0)$, and is such that

$$27G(y_0, u, v) = 16v^3 - 18uv - 27v + 2(4v^2 - 3u)^{3/2}.$$

Hence, $G(y_0, u, v) < 0$, so that $G(y, u, v) < 0$ when $-1 \leq y \leq 0$ and $N(u, v) = 1$, if and only if

$$2(4v^2 - 3u)^{3/2} < 27v + 18uv - 16v^3.$$

The right hand side of this inequality is greater than $27v + 18(v-1)v - 16v^3 = v(9 + 18v - 16v^2)$, and this is positive because $0 < v < 2/3$. Therefore, $G(y_0, u, v) < 0$ in case

$$4(4v^2 - 3u)^3 < (27v + 18uv - 16v^3)^2,$$

and this inequality holds if and only if,

$$R(u, v) = u^3 - u^2v^2 + 9uv^2 + v^2(6.75 - 8v^2) > 0.$$

We see that $3(\partial R/\partial u) = 3(3u^2 - 2v^2u + 9v^2) = (3u - v^2)^2 + v^2(27 - v^2) > 0$ when $v^2 < 27$. Moreover, $R(0, v) = v^2(6.75 - 8v^2) > 0$ when $0 < v < 2/3$, and $R(v-1, v) = -(3v-2)^2(4v^2+1) < 0$ when $v \neq 2/3$. Therefore, when $0 < v < 2/3$, there exists a unique real number $F(v)$ such that $R(F(v), v) = 0, v-1 < F(v) < 0$. Hence, $R(u, v) > 0$ when $u > F(v)$, and $R(u, v) < 0$ when $u < F(v)$. Accordingly, $G(y_0, u, v) < 0$ when $F(v) < u < (4v^2)/3$ and $G(y_0, u, v) > 0$ when $v - 1 < u < F(v)$. Therefore, $N(u, v) = 1$ when $F(v) < u < (4v^2)/3$ and $0 < v < 2/3$.

When $v - 1 < u < F(v)$, however, $G(y_0, u, v) > 0$, $G(-1, u, v) < 0$, and $G(0, u, v) < 0$. Hence, there are exactly two zeros of G on the interval $(-1, 0)$, and so $N(u, v) = 3$ when $v - 1 < u < F(v)$ and $0 < v < 2/3$.

In order to find $F(v)$, we suppose that $v = c^{3/2}, F = cz$. Then $z^3+pz^2+qz+r = 0$, in which $p = -c^2, q = 9c$, and $r = 6.75 - 8c^3$. Using Cardan's formulas [4, p. 42-44], we see that $z = A + B - (p/3)$, in which $a = (3q - p^2)/9 = c(27 - v^2)/9, b = (9pq - 2p^3 - 27r)/54 = (8v^4 + 540v^2 - 729)/216$, $B^3 = b - (b^2 + a^3)^{1/2} = b - \{(8v^2 + 27)/12\}^{3/2}$, $A = -a/B$. The value of $F(v)$ can be calculated, for a given value of v, from these formulas.

We summarize our calculation of $N(u, v)$ in Table 2.

Table 2. REGIONS OF CONSTANT N

$$
\begin{aligned}
N(u,v) &= 1 \text{ when } 0 < v < \infty \text{ and } u < -v-1, \\
&\quad \text{when } 0 < v < 2/3 \text{ and } u > F(v), \\
&\quad \text{and when } v > 2/3 \text{ and } u > v-1; \\
N(u,v) &= 2 \text{ when } 0 < v < \infty \text{ and } -v-1 < u < v-1; \\
N(u,v) &= 3 \text{ when } 0 < v < 2/3 \text{ and } v-1 < u < F(v).
\end{aligned}
$$

CHAPTER 5

The Calculation of ν_3 and μ_3

To calculate ν_3 and μ_3 we will begin with the indefinite integral,

$$\begin{aligned} K(u,v) &= \int [M(u,v)]^{-3/2} du = \int [8u^2+8v^2+12u+5]^{-3/2} du \\ &= (4u+3)/[2(16v^2+1)(8u^2+8v^2+12u+5)^{1/2}]. \end{aligned}$$

With the help of Table 2, we now see that

$$\begin{aligned} 2^{-7/2}\pi\nu_3 &= \textstyle\int_0^\infty [K(-v-1,v)-K(-\infty,v)]dv + \int_0^{2/3}[K(\infty,v)-K(F(v),v)]dv \\ &\quad + \textstyle\int_{2/3}^\infty [K(\infty,v)-K(v-1,v)]dv + 2\int_0^\infty [K(v-1,v)-K(-v-1,v)]dv \\ &\quad +3\textstyle\int_0^{2/3}[K(F(v),v)-K(v-1,v)]dv. \end{aligned}$$

Therefore, $2^{-7/2}\pi\nu_3 = J_1+J_2+J_3+J_4+J_5$, in which

$$\begin{aligned} J_1 &= \textstyle\int_0^\infty K(\infty,v)\ dv - \int_0^\infty K(-\infty,v)\ dv, \\ J_2 &= -\textstyle\int_0^\infty K(-v-1,v)\ dv, \\ J_3 &= \textstyle\int_0^\infty K(v-1,v)dv, \\ J_4 &= -2\textstyle\int_0^{2/3} K(v-1,v)dv, \\ J_5 &= 2\textstyle\int_0^{2/3} K(F(v),v)dv. \end{aligned}$$

Note that $K(\infty,v) = -K(-\infty,v) = 2^{-1/2}(16v^2+1)^{-1}$, *and* $\int_0^\infty (16v^2+1)^{-1}dv = \pi/8$. Hence $J_1 = (\sqrt{2}\pi)/8$.

If we make the sequence of changes of variables of integration,

$$4v = \xi,\ \xi + 1/2 = (\sqrt{3}/2)\tan\ \theta,\ \ \sin\ (\theta - \pi/3) = t/\sqrt{2},$$

we find that

$$\begin{aligned} J_2 &= \int_0^\infty \frac{(4v+1)\ dv}{2(16v^2+1)(16v^2+4v+1)^{\frac{1}{2}}} \\ &= 2^{-\frac{5}{2}} \int_{-2^{-1/2}}^{2^{-1/2}} (1+t^2)^{-1} dt = 2^{-\frac{3}{2}} \tan^{-1}(2^{-1/2}) \approx 0.2176049379. \end{aligned}$$

Similarly, if

$$4v = \xi,\ \xi - 1/2 = (\sqrt{3}/2)\ \tan\ \theta, \sin\ (\theta + \pi/3) = t/\sqrt{2},$$

we find that

$$J_3 = -2^{-5/2} \int_{2^{-1/2}}^{2^{-1/2}} (1+t^2)^{-1} dt = 0,$$

$$J_4 = 2^{-\frac{3}{2}} \int_{-2^{-1/2}}^{s} (1+t^2)^{-1} dt = 2^{-\frac{3}{2}}[\tan^{-1}(s) - \tan^{-1}(2^{-1/2})] \ \approx\ 0.513885010,$$

in which $s = (11\sqrt{2})/14$. Note that $\tan^{-1}(s) \approx 0.838006592$.

Unfortunately, it does not appear that the integral in J_5 is elementary. Nonetheless, we can use Simpson's rule with sufficiently small interval size to get an accurate approximation to J_5. This approximation was computed by the JAVA computer program listed in table 3. This yields the result that $J_5 \approx 0.3028975974$. We can now use the numerical values of J_1, J_2, J_3, J_4, and J_5 to find that

$$\nu_3 = 4.15779773.$$

Moreover,

$$\begin{aligned}
2^{-7/2}\pi\mu_3 &= \int_0^\infty [K(-v-1,v)-K(-\infty,v)]\,dv+\int_0^{2/3}[K(\infty,v)-K(F(v),v)]\,dv \\
&\quad+\int_{2/3}^\infty [K(\infty,v)-K(v-1,v)]\,dv+4\int_0^\infty [K(v-1,v)-K(-v-1,v)]\,dv \\
&\quad+9\int_0^{2/3}[K(F(v),v)-K(v-1,v)]dv \\
&= J_1+3J_2+3J_3+4J_4+4J_5,
\end{aligned}$$

so that

$$\mu_3 = 4-\frac{8}{\pi}\tan^{-1}(2^{-1/2})+\frac{32}{\pi}\tan^{-1}s+\frac{2^{13/2}}{\pi}J_5 \approx 19.69507562.$$

Consequently,

$$\sigma_3^2=\mu_3-\nu_3^2\approx 2.40779366,$$

and

$$\sigma_3 = 1.551706694.$$

Table 3. JAVA PROGRAM FOR J_5

```
public class ComputationOfJ5 {

public static void main (String [] args) {

double q, v, sum, integralSolution, pv;

double h, prevIntegralSol, difference;

h = 1/12D;

prevIntegralSol = 0;

integralSolution = 0;

difference = 0;

do {

prevIntegralSol = integralSolution;

q = 4.0;

v = h;

sum = 3/(Math.sqrt(5.0));

do {

pv = calculatepv(v);

sum = sum + q * pv;

v = v+h;

q = 6-q;

} while (v < (2/3D) + (h/2D));
```

Table 3 - Continued

```
sum = sum - pv;
integralSolution = h/3 * sum;
h = h/8;
difference = Math.abs (integralSolution - prevIntegralSol);
} while (difference > 1/100000000000D );
System.out.println ("integral solution = " + integralSolution);
}
public static double calculatepv (double v) {
double u = Math.pow (v, 2);
double c = Math.pow (u, 1/3D);
double t = Math.pow (((8*u + 27) / 12), 3/2D);
double d = (c * (27 - u) ) / 9;
double w = (729 - u * (8*u + 540) ) / 216;
double b = - (Math.pow ((w + t), 1/3D) );
double a = - (d / b);
double f = (a + b) * c + (u / 3);
double pv = ((4*f) + 3) / ((16*u+1) * Math.sqrt (f*(8*f + 12) + 5+8*u));
return pv;
}
}
```

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